Analytical Expressions for the REM Model of Recognition Memory

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Abstract

An inordinate amount of computation is required to evaluate predictions of simulation-based models. Following Myung et al (2007), we derived an analytic form expression of the REM model of recognition memory using a Fourier transform technique, which greatly reduces the time required to perform model simulations. The accuracy of the derivation is verified by showing a close correspondence between its predictions and those reported in Shiffrin and Steyvers (1997). The derivation also shows that REM's predictions depend upon the vector length parameter, and that model parameters are not identifiable unless one of the parameters is fixed.

1 Introduction

Much of the recent formal modeling work in recognition memory is simulation-based (Shiffrin et al., 1990; Shiffrin and Steyvers, 1997; Hintzman, 1988; McClelland and Chappell, 1998; Dennis and Humphreys, 2001). A simulation-based model is defined in terms of algorithmic steps that detail how the specific latent mental processes, such as memory storage, memory retrieval and a recognition decision, interact with one another to yield an observable response. The assumed processes are often too complicated to express model predictions in an analytic form. Instead, the algorithmic steps involved must be simulated on computer with the help of random number generators to derive predictions. This is a time-consuming task because a large number of simulation replications often need to be run to obtain an accurate estimate of the model's prediction.

The excessive computing time required to derive predictions in simulation-based models has made it a major challenge to identify the best-fitting parameter values given observed data, as well as generate the expected pattern of model behavior as the values of model parameters change. This in turn has made it difficult to assess the descriptive adequacy of one particular simulation-based model and to evaluate it relative to alternative simulation-based models of the same phenomenon.

To combat this challenge, we introduced a Fourier transform (FT) technique that allows one to derive closed-form expressions for simulation-based models of recognition memory (Myung et al., 2007). The resulting expressions are given in the form of integral equations that still need to be numerically evaluated using a computer, but are generally much easier and faster to compute than a brute-force model simulation.

Myung et al. (2007) applied the FT technique to the Bind Cue Decide Model of Episodic Memory (BCDMEM; Dennis and Humphreys, 2001) and derived one-dimensional integral equations that enable one to readily compute hit and false alarm probabilities for any given values of the model's parameters. An unanticipated by-product of the approach was that it revealed properties of the model that were not apparent otherwise. One such property that we were able to glean from the asymptotic expressions was that the model with its five parameters is unidentifiable, and as such, there exist infinitely many different sets of model parameters that all provide equally good fits to observed data summarized as hit and false alarm (FA) rates. Particularly noteworthy is the observation that the vector length parameter is not an ignorable parameter, as had been thought previously, but instead it can significantly affect model predictions. An obvious next question one could then ask is whether the FT technology could be used to derive integral expressions for other simulation-based models of recognition memory.

We report such a derivation for the Retrieving Effectively from Memory (REM) model of recognition memory (Shiffrin and Steyvers, 1997). Specifically, we derived analytic expressions in a double-integral form for hit and false alarm probabilities that the model predicts given its parameter values. A close examination of the results shows it too possesses some of the same properties as BCDMEM: The model is unidentifiable unless one of its parameters is fixed to a predetermined value, and the vector length parameter is not an ignorable parameter. We begin with a short review of the assumptions of REM before presenting the FT-based derivation of integral expressions for recognition probabilities of the model.

2 REM

REM is a model for human performance in recognition memory tasks. In a typical experiment, participants are first asked to study a list of items, such as words or syllables, and are then later tested for how well they discriminate old items (i.e., shown in the study phase) from new items (i.e., not studied). Specifically, the model assumes that each word is stored separately in memory as a multi-dimensional vector of w feature values made of non-negative integers (e.g., $(1,0,7,3)$), such that the value of 0 denotes the absence of knowledge about a feature whereas non-zero values denote knowledge of a feature and are generated from a geometric distribution with probability parameter g. Specifically, the probability of generating a particular feature value j is given by $P(j) = g(1-g)^{j-1}, j = 1, 2, ..., \infty$. The model further assumes that recognition memory performance consists of a series of three processes that follow one after another: memory storage, memory retrieval, and recognition decision.

The memory storage process is assumed to be imperfect in that on each unit of time when an item is studied, a feature is attended and subsequently stored with probability u. Once attended, the feature value is copied correctly into memory with probability c, or a random value is copied into memory with probability $(1-c)$. The model assumes that experimental manipulations of stimulus presentation determine the units of storage time. For example, slow or multiple presentations would correspond to higher unit values, whereas fast or single presentations would result in lower unit values. In sum, the model assumes four parameters, with (g, u, c) capturing memory processes and w denoting the vector length.

Second, in memory retrieval, upon the presentation of a probe (test) item, the probe vector is compared against each stored vector in memory, feature by feature, resulting in counts of match and mismatch between the probe and stored vectors. From the match/mismatch counts given a stored vector, the likelihood ratio or odds, λ, is defined as the ratio of the probability of observing the probe item under the hypothesis that it is old (i.e., the stored vector is a possibly imperfect representation of the probe) relative to the probability of observing the probe under the hypothesis that it is new (i.e., the stored vector has no relation to the probe):

$$\lambda\left(n_a, n_{m1}, ..., n_{m\infty}|(g,u,c),w\right) = (1-c)^{n_a} \prod_{i=1}^{\infty} \left(\frac{c + (1-c)g(1-g)^{i-1}}{g(1-g)^{i-1}} \right)^{n_{mi}}. \tag{1}$$

In the equation, n_a is the number of mismatching nonzero features, restricted to non-zero values on the stored vector and regardless of value, and n_{mi} is the number of matching nonzero features with value i $(= 1, 2, ..., \infty)$. It is worth noting that zero feature values on the stored vector are not utilized in the above equation as they provide no differential evidence for a recognition decision (Shiffrin and Steyvers, 1997, p. 163).

The model assumes that the recognition decision is based on the arithmetic average of λ's over all stored vectors in memory, defined as

$$\Phi = \frac{1}{N} \sum_{j=1}^{N} \lambda_j (n_a^j, n_{m1}^j, ..., n_{m\infty}^j | (g,u,c), w), \tag{2}$$

where the the list length N denotes the number of words in the studied list. The decision of 'old' or 'new' is then made such that

$$\begin{array}{lcl} \Phi > 1 & \Longrightarrow & \text{decide "old"} \\ \Phi < 1 & \Longrightarrow & \text{decide "new."} \end{array} \tag{3}$$

3 Integral Expressions for Recognition Probabilities

In this section we derive analytic, integral expressions of hit and false alarm (FA) rates that REM predicts given a specific set of parameter values. The derivation below follows many of the same steps for deriving the Fourier integral expressions for BCDMEM (Myung et al., 2007, pp. 199-202).

3.1 Feature Matching as a Multinomial Process

Given that features are sampled independent of one another during memory retrieval, the matching and mismatching counts between a probe vector and the j-th stored memory vector, i.e., $(n_a^j, n_{m1}^j, ..., n_{m\infty}^j)$ in Eq. (1), are samples from a multinomial probability distribution defined on a discrete random variable. Although a zero-valued feature contains no differential information, for the sake of completeness, however, let n_0^j denote the number of zero-valued features of the jth stored vector and define $\boldsymbol{n}^j = (n_0^j, n_a^j, n_{m1}^j, ..., n_{m\infty}^j)$ with $\sum_\nu n_\nu^j = w$ for each j $(= 1, ..., N)$, where N is the number of studied items during study and is often called the *list length* in the memory literature, $\nu = (0, a, m1, m2, ..., m\infty)$ is an index variable, and finally, w is the total number of features or the vector length parameter.

Before we derive the expression for the multinomial probability distribution of the matching and mismatching counts (i.e., $\boldsymbol{n}^j$), we make a distinction between two types of random processes pertaining the parameter g – a distinction made in Shiffrin and Steyvers (1997, Fig. 1). From this point on, we will use the asterisk symbol g^* to denote the parameter of a geometric distribution from which non-zero feature values of actual words used in the study list are generated, and then use the non-asterisk symbol g to denote the long-run environmental base rate of non-zero feature values used to calculate the likelihood ratio λ_j in Eqs. (1)-(2) given the particular values of $\boldsymbol{n}^j$ in order to make the recognition decision in Eq. (3). It is important to note that the probability distribution of $\boldsymbol{n}^j$ is determined by the parameter g^* as well as parameters (u, c, w), but is independent of the long-run base rate parameter g.

From the memory storage and retrieval assumptions of REM and further, under the assumption that features are independent of one another, it is straightforward to show that the probability distribution of the matching and mismatching counts between a probe item that is old (i.e., presented during study) and the retrieved image of the probe from memory, which we refer to as the Nth stored vector, without loss of generality, is given by

$$f(\boldsymbol{n}^N|\boldsymbol{p}(g^*, u, c), w) = w! \prod_{\nu} \frac{p_\nu(g^*, u, c)^{n_\nu^N}}{n_\nu^N!}. \tag{4}$$

In the above equation, $\boldsymbol{p}(g^*, u, c) = \{p_\nu(g^*, u, c); \sum_\nu p_\nu(g^*, u, c) = 1\}$ is a vector consisting of multinomial probability parameters, each corresponding to a matching or mismatching type, obtained as

$$\begin{aligned} p_0(g^*, u, c) &= 1 - u^* \\ p_a(g^*, u, c) &= 2u^*(1-c)\frac{(1-g^*)}{(2-g^*)} \\ p_{mi}(g^*, u, c) &= u^*\left(c + (1-c)g^*(1-g^*)^{i-1}\right)g^*(1-g^*)^{i-1}, \quad (i = 1, ..., \infty), \end{aligned} \tag{5}$$

where $u^* = 1 - (1-u)^t$ is the probability of generating a non-zero feature value after t (=1,2,...) storage attempts, with $u^* = u$ per storage attempt (i.e., $t = 1$). That is, the model assumes that multiple attempts are made to store in memory each feature of each item.

The probability distribution of the matching and mismatching counts between an old probe item and each of the remaining $(N-1)$ stored vectors is given by

$$f(\boldsymbol{n}^j|\boldsymbol{q}(g^*, u, c), w) = w! \prod_{\nu} \frac{q_\nu(g^*, u, c)^{n_\nu^j}}{n_\nu^j!}, \quad (j = 1, ..., N-1), \tag{6}$$

with

$$\begin{aligned} q_0(g^*, u, c) &= 1 - u^* \\ q_a(g^*, u, c) &= 2u^*\frac{(1-g^*)}{(2-g^*)} \\ q_{mi}(g^*, u, c) &= u^*\left(g^*(1-g^*)^{i-1}\right)^2, \quad (i = 1, ..., \infty), \end{aligned} \tag{7}$$

and $\sum_\nu q_\nu(g^*, u, c) = 1$. Note that the same set of parameter values (g^*, u, c) are assumed to be used to generate all N stored vectors in memory.

Putting together the results in Eqs. (4)-(7) and assuming that stored items are independent, we obtain the probability distribution of all matching and mismatching counts for an old probe item as

$$\begin{aligned} & f(\boldsymbol{n}^1, ..., \boldsymbol{n}^N | \boldsymbol{p}(g^*, u, c), \boldsymbol{q}(g^*, u, c), w) \\ = \; & f(\boldsymbol{n}^N | \boldsymbol{p}(g^*, u, c), w) \prod_{j=1}^{N-1} f(\boldsymbol{n}^j | \boldsymbol{q}(g^*, u, c), w) \\ = \; & \left(w! \prod_{\nu} \frac{p_\nu(g^*, u, c)^{n_\nu^N}}{n_\nu^N!} \right) \left(\prod_{j=1}^{N-1} w! \prod_{\nu} \frac{q_\nu(g^*, u, c)^{n_\nu^j}}{n_\nu^j!} \right). \end{aligned} \tag{8}$$

Similarly, the matching and mismatching counts for a new item that was *not* presented during study follow another multinomial probability distribution of the form:

$$h(\boldsymbol{n}^1, ..., \boldsymbol{n}^N | \boldsymbol{q}(g^*, u, c), w) = \prod_{j=1}^{N} w! \prod_{\nu} \frac{q_\nu(g^*, u, c)^{n_\nu^j}}{n_\nu^j!}. \tag{9}$$

Before closing the present section, we note that in terms of the matching and mismatching counts n_ν^j, the likelihood ratio λ_j in Eq. (1) for the j-th stored vector is re-expressed as

$$\lambda_j(\boldsymbol{n}^j | \, \theta, w) = \exp\left(\sum_{\nu} \beta_\nu(g, u, c) \, n_\nu^j \right), \tag{10}$$

with

$$\begin{aligned} \beta_0(g, u, c) &= 0 \\ \beta_a(g, u, c) &= \ln(1 - c) \\ \beta_{mi}(g, u, c) &= \ln\left(\frac{c + (1 - c) g (1 - g)^{i-1}}{g (1 - g)^{i-1}} \right). \end{aligned} \tag{11}$$

As mentioned earlier, it is worth noting that the distribution of n_ν^j in Eq. (10) depends on the parameter g^* with which the actual words are generated, but not on the environmental base rate g being used to make a recognition decision. Further, as will be seen later, the linear relationship in Eq. (10) plays a crucial role in obtaining integral expressions of model probabilities. Finally, the mean odds Φ in Eq. (2) that determines a recognition response of "old" or '"new" is rewritten as

$$\Phi(\boldsymbol{n}^1, ..., \boldsymbol{n}^N | (g, u, c), w) = \frac{1}{N} \sum_{j=1}^{N} \exp\left(\sum_{\nu} \beta_\nu(g, u, c) \, n_\nu^j \right), \tag{12}$$

which we use to express hit and FA rates of REM in the following section.

3.2 Hit and FA Rates as Multinomial Expectations

We are now ready to express two primary predictions of REM, hit and FA rates as expectations under appropriate multinomial distributions, similarly as was done for BCDMEM (Myung et al., 2007, p. 200). The hit rate, which is expressed formally as $P(Hit|\,\theta = (g, g^*, u, c), w)$ in terms of a four-element parameter vector θ, is the probability of correctly deciding "old" for an old probe (test) item and is therefore an expectation under the multinomial distribution in Eq. (4):

$$P(\text{Hit}|\,\theta = (g, g^*, u, c), w) = \sum_{\sum_\nu n^1_\nu = w} \cdots \sum_{\sum_\nu n^N_\nu = w} S\left(\Phi(\boldsymbol{n}^1, ..., \boldsymbol{n}^N|\phi, w) - 1\right) f(\boldsymbol{n}^1, ..., \boldsymbol{n}^N|\boldsymbol{p}(\phi^*), \boldsymbol{q}(\phi^*), w), \tag{13}$$

where $\phi = (g, u, c), \phi^* = (g^*, u, c)$, and $S(x)$ is the step function defined as $S(x) = 1$ if $x > 0$ and 0 otherwise. Similarly, the FA rate, the probability of incorrectly deciding "old" for a new probe item, is given by

$$P(\text{FA}|\,\theta = (g, g^*, u, c), w) = \sum_{\sum_\nu n^1_\nu = w} \cdots \sum_{\sum_\nu n^N_\nu = w} S\left(\Phi(\boldsymbol{n}^1, ..., \boldsymbol{n}^N|\phi, w) - 1\right) h(\boldsymbol{n}^1, ..., \boldsymbol{n}^N|\boldsymbol{q}(\phi^*), w). \tag{14}$$

Predicting hit and FA rates for each chosen set of parameter values requires the evaluation of the massive summations in Eqs. (13)-(14), exact calculations of which are generally difficult and computationally expensive, especially for large N and w. This is because of the combinatorial explosion problem in which the number of individual terms to be summed across increases rapidly with w (Myung et al., 2007, footnote 1). Given the intractability of an exact solution, one must resort to brute-force methods in which for a specific set of parameter values, the model is simulated on computer. This process is repeated a large number of times to ensure that accurate estimates of hit and FA rates are obtained, which is a time-consuming and inefficient means of modeling. Next we show how the hit and FA rates in Eqs.(13)-(14) can be much easier to compute when the double-integral forms are rewritten analytically using the Fourier transformation technique.

3.3 Fourier Integral Expressions

Fourier transformation (FT) is a mathematical tool for re-expressing a real-valued function as an infinite integral of complex-valued sinusoidal functions. The step function $S(x)$ in Eq. (13), for example, takes the following equivalent, FT form:

$$S(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{ik} \;\; dk \tag{15}$$

where e^{ikx} is defined as $\cos(kx) + i\sin(kx)$ and $i = \sqrt{-1}$.[1]

Plugging the above FT form and the expressions in Eqs. (10)-(12) into Eq. (13), we obtain

[1]Eq. (15)is an "informal" form of the FT for the step function. The full and formal FT form is defined as a limiting integral representation $S(x) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k - i\epsilon} dk$ so as to avoid the singularity at $k = 0$ (e.g., *Wikipedia*: http://en.wikipedia.org/wiki/Heaviside step function). For this expression, it can be shown that $S(-x) = 1 - S(x)$.

an equivalent expression for $P(\text{Hit}|\,\theta = (g, g^*, u, c), w)$ as

$$
\begin{aligned}
& P(\text{Hit}|\,\theta, w) \\
= \;& \sum_{\sum_\nu n_\nu^1 = w} \cdots \sum_{\sum_\nu n_\nu^N = w} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{ik} e^{ik\left(\frac{1}{N}\sum_{j=1}^N \lambda_j(\boldsymbol{n}^j|\,\phi, w) - 1\right)} dk \right] f(\boldsymbol{n}^1, ..., \boldsymbol{n}^N | \boldsymbol{p}(\phi^*), \boldsymbol{q}(\phi^*), w) \\
= \;& \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \frac{e^{-ik}}{ik} \sum_{\sum n_\nu^1 = w} \cdots \sum_{\sum n_\nu^N = w} e^{\frac{ik}{N}\sum_j \lambda_j(\boldsymbol{n}^j|\,\phi, w)} f(\boldsymbol{n}^1, ..., \boldsymbol{n}^N | \boldsymbol{p}(\phi^*), \boldsymbol{q}(\phi^*), w) \qquad (16) \\
= \;& \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \frac{e^{-ik}}{ik} \left[\sum_{\sum n_\nu^N = w} e^{\frac{ik}{N}\lambda_N(\boldsymbol{n}^N|\,\phi, w)} f(\boldsymbol{n}^N | \boldsymbol{p}(\phi^*), w) \right] \left[\prod_{j=1}^{N-1} \sum_{\sum n_\nu^j = w} e^{\frac{ik}{N}\lambda_j(\boldsymbol{n}^j|\,\phi, w)} f(\boldsymbol{n}^j | \boldsymbol{q}(\phi^*), w) \right] \\
= \;& \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \frac{e^{-iNk}}{ik} \left[\sum_{\sum n_\nu^N = w} e^{ik\lambda_N(\boldsymbol{n}^N|\,\phi, w)} f(\boldsymbol{n}^N | \boldsymbol{p}(\phi^*), w) \right] \left[\prod_{j=1}^{N-1} \sum_{\sum n_\nu^j = w} e^{ik\lambda_j(\boldsymbol{n}^j|\,\phi, w)} f(\boldsymbol{n}^j | \boldsymbol{q}(\phi^*), w) \right]
\end{aligned}
$$

where the second last equality is derived from Eq. (8) and the last equality is obtained by the variable transformation $k' = k/N$. Note that the expressions in the hard brackets [...] represent the expectations under the appropriate multinomial probability distribution. Since we are dealing with expected values of $e^{ik\lambda_j}, (j = 1, ..., N)$, that are independent of the word index j, we can write

$$
\begin{aligned}
& P(\text{Hit}|\,\theta, w) \\
= \;& \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \frac{e^{-iNk}}{ik} E\left[e^{ik\lambda(\boldsymbol{n}|\,\phi, w)} |\, f(\boldsymbol{n}|\boldsymbol{p}(\phi^*), w) \right] \left(E\left[e^{ik\lambda(\boldsymbol{n}|\,\phi, w)} |\, f(\boldsymbol{n}|\boldsymbol{q}(\phi^*), w) \right] \right)^{N-1}. \qquad (17)
\end{aligned}
$$

A separate application of FT allows one to rewrite the multinomial expectation as the following equivalent FT form (see the Appendix):

$$
\begin{aligned}
E\left[e^{ik\lambda(\boldsymbol{n}|\,\phi, w)} |\, f(\boldsymbol{n}|\boldsymbol{r}(\phi^*), w) \right] &= 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dz}{iz} \, \Gamma(1 - iz)(-ik)^{iz} \left(\sum_\nu e^{iz\beta_\nu(\phi)} r_\nu(\phi^*) \right)^w \\
&\equiv G_w(k, \boldsymbol{r}(\phi^*), \boldsymbol{\beta}(\phi)). \qquad (18)
\end{aligned}
$$

In the above equation, $\boldsymbol{r}(\phi^*) = \boldsymbol{p}(\phi^*)$ or $\boldsymbol{q}(\phi^*)$ with $\phi^* = (g^*, u, c)$, z is the dummy integration variable, $\Gamma(\cdot)$ is the Gamma function defined as $\Gamma(x) = \int_0^\infty e^{-z} z^{x-1} \, dz$, and finally, $\boldsymbol{\beta}(\phi) = \{\beta_\nu(\phi)\}$ with $\phi = (g, u, c)$ is defined in Eq. (11).

Putting the results in Eqs. (17)-(18) together, we now obtain the desired expressions for the hit rate

$$
P(\text{Hit}|\,\theta, w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \frac{e^{-iNk}}{ik} \, G_w(k, \boldsymbol{p}(\phi^*), \boldsymbol{\beta}(\phi)) \, G_w(k, \boldsymbol{q}(\phi^*), \boldsymbol{\beta}(\phi))^{N-1}, \qquad (19)
$$

and similarly, for the false alarm rate

$$
P(\text{FA}|\,\theta, w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \frac{e^{-iNk}}{ik} \, G_w(k, \boldsymbol{q}(\phi^*), \boldsymbol{\beta}(\phi))^N, \qquad (20)
$$

rewritten as Fourier integral functions of model parameters, $\theta = (g, g^*, u, c), w$, and N.

3.4 Verification of FT Expressions

The Fourier integral expressions for REM in Eqs. (19)-(20) are similar to those obtained for BCDMEM (Myung et al., 2007, Eq. 10) in that in both cases, the recognition probability, hit or false alarm, is shown to be a line integration along the real axis in the complex plane with a singularity at $k = 0$. The residue theorem of complexity analysis (Arfken and Weber, 2001) allows one to convert the integration to a simple integral of a real-valued function along the real line (Myung et al., 2007, Eq. 11), which can then be evaluated numerically using *Matlab*, for example.

Numerical computation for REM, however, is expected to be significantly more challenging than that for BCDMEM. This is because the integrals in Eqs. (19)-(20) contain an integrand function $G_w(k, \boldsymbol{r}(\phi^*), \boldsymbol{\beta}(\phi))$, where $\boldsymbol{r}(\phi^*) = \boldsymbol{p}(\phi^*)$ or $\boldsymbol{q}(\phi^*)$, which itself takes the form of another line integration on the real axis of the complex plane with a singularity at the origin. This is not the case for BCDMEM, in which the integrand function is a simple real-valued function. In short, the integral expressions in Eqs. (19)-(20) need to be calculated with care and attention given to the convergence and consistency of the numerical solution obtained.

The verification of the correctness of the FT integral expressions of recognition probabilities was performed in two ways. First, we attempted to replicate the simulation results reported in Shiffrin and Steyvers (1997). In their simulation of recognition memory, the authors represented each word by 20 features ($w = 20$). The features were generated from a geometric distribution with probability parameter $g^* = 0.45$ and were then stored in memory with $u = 0.04$ per storage attempt, $t = 10$ storage attempts (i.e., $u^* = 1 - (1 - u)^{10} = 0.335$), and $c = 0.70$. The multinomial feature matching process was performed assuming $g = 0.4$.[2] The list length was varied across five different values of $N = 4, 10, 20, 40, 80$. Their results are shown in the third column of Table 1, and our own simulation results are shown in the fourth through seventh columns for different values of the vector length parameter (w). Note that for $w = 20$, our values match closely those of Shiffrin and Steyvers (1997), thereby verifying the correctness of our implementation of REM.

Second, we also assessed the correspondence between the two forms of model predictions, direct model simulation and FT-based analytic calculation. In Figure 1, the solid circles represent the simulation predictions of Shiffrin and Steyvers (1997) and the dotted lines represent the FT-based predictions. Note that the values of the analytic FT integrals almost exactly match those of the *brute-force* simulations. Taken together, these results demonstrate the correctness of the FT expressions.

Finally, as found with BCDMEM, parameter interpretability can be an issue with REM. The integral expressions in Eqs. (19)-(20) make it clear that model predictions depend upon the vector length parameter (w), through the function $G_w(\cdot)$, which appears in both equations. This means that the memory vector length is a non-ignorable parameter and can strongly affect model predictions. In particular, the simulation results in Table 1 suggest that as w goes to infinity, the hit and FA rates converge to 1 or 0, respectively, presumably for all values of the other three parameters $\theta = (g, g^*, u, c)$, which represent the putative memory processes. This means that REM is not identifiable: There exist many sets of parameter values that make exactly the same model predictions (i.e., hit and false alarm rates). An implication is that the three *memory-process* parameters may not be meaningfully interpretable unless vector length is fixed, as in Shiffrin and Steyvers (1997) (i.e., $w = 20$).

[2]Recall that $g^* = 0.45$ is the parameter value actually used to generate an item vector whereas $g = 0.40$ is the parameter value used to simulate the recognition decision in Eq. (3).

4 Conclusion

This theoretical note contributes a Fourier integral expression of recognition probabilities for REM that can be evaluated analytically, thereby avoiding the time-consuming task of brute-force simulation of the model as originally formulated. Recently, Turner and colleagues (Turner and Van Zandt, 2012; Turner and Sederberg, 2012) demonstrated the application and promise of Approximate Bayesian Computation (ABC: Beaumont et al., 2002; Sisson et al., 2007) for making Bayesian inferences for simulation-based models of cognition, including REM. ABC is a sampling-based method of posterior estimation that does not require a closed-form likelihood. Although currently in its infancy, if the method proves to be reliable, it could possibly replace the FT-based technique.

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Appendix

This appendix shows the main steps of the derivation of Eq. (18). We are interested in evaluating an expectation defined in the following form

$$E\left[e^{ik\lambda(\boldsymbol{n}|\,\phi,w)}|\, f(\boldsymbol{n}|\boldsymbol{r}(\phi^*),w)\right] = \sum_{\sum n_\nu = w} e^{ik\lambda(\boldsymbol{n}|\,\phi,w)}\, f(\boldsymbol{n}|\boldsymbol{r}(\phi^*),w). \tag{21}$$

where $\boldsymbol{r}(\phi^*) = \boldsymbol{p}(\phi^*)$ or $\boldsymbol{q}(\phi^*)$. For the sake of simplicity, for now, we will drop the conditional notation with reference to $f(\cdot)$ in the hard bracket [...] on the left-hand side of the above equation.

Using the FT expression of the step function $S(x)$ in Eq. (15), we first note the following equalities:

$$\begin{aligned} E\left[S\left(\lambda(\boldsymbol{n}|\,\phi,w)-\eta\right)\right] &= \sum_{\sum n_\nu = w}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{ik}e^{ik(\lambda(\boldsymbol{n}|\,\phi,w)-\eta)}\right) f(\boldsymbol{n}|\boldsymbol{r}(\phi^*),w) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{ik}e^{-ik\eta}\sum_{\sum n_\nu = w} e^{ik\lambda(\boldsymbol{n}|\,\phi,w)}\, f(\boldsymbol{n}|\boldsymbol{r}(\phi^*),w) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{ik}e^{-ik\eta}\, E\left[e^{ik\lambda(\boldsymbol{n}|\,\phi,w)}\right]. \end{aligned} \tag{22}$$

The last equality of the above equation can be interpreted as a Fourier transformation between the two functions expressed as expectations. As such, the desired expectation can now be written as

$$
\begin{aligned}
E\left[e^{ik\lambda(\boldsymbol{n}|\,\phi,w)}\right] &= ik\int_{-\infty}^{\infty} d\eta\, e^{ik\eta}\, E\left[S\left(\lambda(\boldsymbol{n}|\,\phi,w)-\eta\right)\right] \\
&= ik\int_{-\infty}^{\infty} d\eta\, e^{ik\eta}\, E\left[S\left(\ln\lambda(\boldsymbol{n}|\,\phi,w)-\ln\eta\right)\right] \\
&= 1+ik\int_{0}^{\infty} d\eta\, e^{ik\eta}\, E\left[S\left(\ln\lambda(\boldsymbol{n}|\,\phi,w)-\ln\eta\right)\right].
\end{aligned} \tag{23}
$$

In the above equation, the second last equality is derived from the fact that the logarithmic function is a monotonically increasing function and thus $\lambda(\boldsymbol{n}|\,\phi,w) > \eta \Leftrightarrow \ln\lambda(\boldsymbol{n}|\,\phi,w) > \ln\eta$ for $\eta > 0$, and further, the last equality is derived from first splitting the integral into positive and negative values of η and evaluating each portion separately.

The new expectation $E\left[S\left(\ln\lambda(\boldsymbol{n}|\,\phi,w)-\ln\eta\right)\right]$ in Eq. (23) can be easily calculated because the logarithmic function linearizes the decision criterion in Eq. (10), i.e., $\ln\lambda(\boldsymbol{n}|\,\phi,w) = \sum_\nu \beta_\nu(\phi)\, n_\nu$. With this linearized expression and also the FT form in Eq. refFT), we obtain

$$
\begin{aligned}
E\left[S\left(\ln\lambda(\boldsymbol{n}|\,\phi,w)-\ln\eta\right)\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{ik}e^{-ik\ln\eta}\sum_{\sum n_\nu = w} e^{ik\sum_\nu \beta_\nu(\phi)\, n_\nu}\, f(\boldsymbol{n}|\boldsymbol{r}(\phi^*),w) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{ik}e^{-ik\ln\eta}\left(\sum_\nu e^{ik\beta_\nu(\phi)} r_\nu(\phi^*)\right)^w.
\end{aligned} \tag{24}
$$

Plugging the above result into Eq. (23) yields

$$
\begin{aligned}
E\left[e^{ik\lambda(\boldsymbol{n}|\,\phi,w)}\right] &= 1+ik\int_{0}^{\infty} d\eta\, e^{ik\eta}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dz}{iz}e^{-iz\ln\eta}\left(\sum_\nu e^{iz\beta_\nu(\phi)} r_\nu(\phi^*)\right)^w \\
&= 1+ik\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dz}{iz}\left(\sum_\nu e^{iz\beta_\nu(\phi)} r_\nu(\phi^*)\right)^w \int_0^\infty d\eta\, e^{ik\eta}\,\eta^{-iz} \\
&= 1+ik\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dz}{iz}\left(\sum_\nu e^{iz\beta_\nu(\phi)} r_\nu(\phi^*)\right)^w \frac{\Gamma(1-iz)}{(-ik)^{1-iz}}
\end{aligned} \tag{25}
$$

where the last equality is derived using the Laplace transformation of a power of η defined as $\int_0^\infty d\eta\, e^{-u\eta}\eta^v \equiv \frac{\Gamma(v+1)}{u^{v+1}}$ for $u = -ik$ and $v = -iq$. By simplifying the above equation, we finally arrive at the desired expectation as follows:

$$
E\left[e^{ik\lambda(\boldsymbol{n}|\,\phi,w)}|\, f(\boldsymbol{n}|\boldsymbol{r}(\phi^*),w)\right] = 1-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dz}{iz}\,\Gamma(1-iz)(-ik)^{iz}\left(\sum_\nu e^{iz\beta_\nu(\phi)} r_\nu(\phi^*)\right)^w \tag{26}
$$

where the right-hand side of the equation defines $G_w(k,\boldsymbol{r}(\phi^*),\boldsymbol{\beta}(\phi))$ in Eq. (18).

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Figure Caption

Figure 1. The dotted curves were obtained by analytically evaluating the Fourier integral expressions. The solid circles represent model simulations from Shiffrin and Steyvers (1997, Fig. 3), as shown in the third column of Table 1. The parameter values in both cases were $g = 0.40, u^* = 0.04, t = 10, c = 0.70, g^* = 0.45$, and $w = 20$.

Table 1: The third column shows model simulations reported in Shiffrin and Steyvers (1997, Fig. 3). The probability values were estimated from a digital copy of the published figure. The fourth through seventh columns show the simulation results. Parameter values used in the simulations were $g = 0.40, u^* = 0.04, t = 10, c = 0.70$, and $g^* = 0.45$. The vector length (w) and list length (N) were varied as shown.

		Shiffrin & Steyvers (1997)	Simulated Values			
Vector length (w)		20	10	20	50	500
List length (N)						
4	Hit	0.861 ± 0.030	0.756	0.864	0.980	1
	FA	0.117 ± 0.030	0.218	0.122	0.021	0
10	Hit	0.812 ± 0.030	0.683	0.816	0.966	1
	FA	0.142 ± 0.025	0.252	0.144	0.028	0
20	Hit	0.779 ± 0.025	0.644	0.775	0.959	1
	FA	0.168 ± 0.025	0.274	0.164	0.038	0
40	Hit	0.731 ± 0.025	0.609	0.733	0.946	1
	FA	0.202 ± 0.025	0.302	0.193	0.042	0
80	Hit	0.694 ± 0.025	0.581	0.694	0.929	1
	FA	0.223 ± 0.025	0.332	0.210	0.052	0

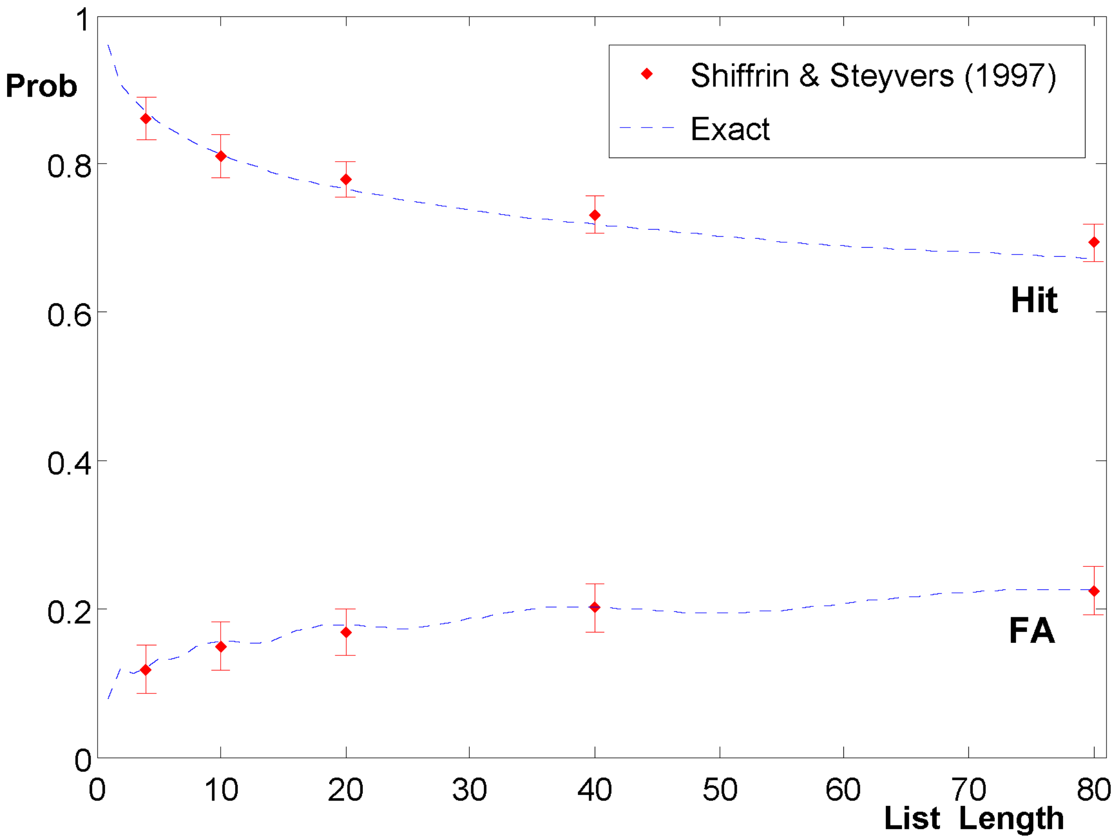

Figure 1: The dotted curves were obtained by analytically evaluating the Fourier integral expressions. The solid circles represent model simulations from Shiffrin and Steyvers (1997, Fig. 3), as shown in the third column of Table 1. The parameter values in both cases were $g = 0.40, u^* = 0.04, t = 10, c = 0.70, g^* = 0.45$, and $w = 20$.